On the Open String Ending on D-brane

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We obtain background independent solutions for an open string ending on D-brane, in variable external fields. Explicit solution of the boundary conditions is given for background metric and NS-NS two-form gauge field, depending on the coordinates of the transverse to the Dp-brane directions. Extension of the constraint algebra is proposed and discussed from both Hamiltonian and Lagrangian approach viewpoint.

1 Introduction

Obtaining exact solutions of the nonlinear probe string equations of motion and constraints in variable external fields is by all means an interesting task with many possible applications. One such application is connected with the recent investigations of the open string - D-brane system in non-constant background fields [1]-[8], in which noncommutative Yang-Mills and noncommutative open string theories can arise on the D-brane worldvolume. Of course, in this case, one is forced to use different approximations in order to get explicit results. That is why, it is interesting to see to what extent our knowledge about the existing *exact* string solutions can help us in considering this dynamical system.

In this paper, we obtain background independent solutions of the open string equations of motion and constraints in non-constant background metric and NS-NS two-form gauge field. Explicit solution of the boundary conditions for the open string ending on D-brane is given for background metric and NS-NS fields, depending on the coordinates of the transverse to the Dp-brane directions. Then we check on an example their compatibility with the solutions of the equations of motion and constraints. After that, we reinterpret the conditions for existence of such solutions as a set of constraints and compute their Poisson bracket algebra. The consequences from the Lagrangian approach viewpoint are also given.

2 The Open String - D-brane System in String Theory Background

The action for an open string ending on a Dp-brane, in the presence of background gravitational and NS-NS 2-form field, can be written as

$$\begin{aligned} S_1 &= -\frac{T}{2}\int d^2\xi\left[\sqrt{-\gamma}\gamma^{mn}\partial_m X^M\partial_n X^N g_{MN}(X)\right. \qquad (1)\\ &- \left.\varepsilon^{mn}\partial_m X^M\partial_n X^N B_{MN}(X)\right] - \frac{T}{2}\int d^2\xi\varepsilon^{mn}\partial_m Y^\mu\partial_n Y^\nu F_{\mu\nu}(Y),\\ &\partial_m = \partial/\partial\xi^m, \quad \xi^m = (\xi^0,\xi^1) = (\tau,\sigma),\\ &m,n = 0,1, \quad M,N = 0,1,...,D-1,\\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu,\nu = 0,1,...,p, \end{aligned}$$

where $T = (2\pi\alpha')^{-1}$ is the (fundamental) string tension,

$$G_{mn}(X) = \partial_m X^M\partial_n X^N g_{MN}(X), \quad B_{mn}(X) = \partial_m X^M\partial_n X^N B_{MN}(X)$$

are the pullbacks of the background metric and antisymmetric NS-NS tensor to the string worldsheet, γ is the determinant of the auxiliary metric γ_{mn}, $Y^\mu(\xi)$ are the coordinates on the D-brane, and A_μ is the $U(1)$ gauge field living on the D-brane worldvolume. In *static gauge* for the D-brane, one identifies Y^μ with the string embedding coordinates $X^\mu(\xi)$. Then the action (1) acquires the form

$$\begin{aligned} S_2 = &- \frac{T}{2}\int d^2\xi\left[\sqrt{-\gamma}\gamma^{mn}\partial_m X^M\partial_n X^N g_{MN}(X)\right. \qquad (2)\\ &- \left.\varepsilon^{mn}\partial_m X^M\partial_n X^N B'_{MN}(X)\right], \end{aligned}$$

where

$$B'_{MN} = B_{MN} - \delta^\mu_M\delta^\nu_N F_{\mu\nu}.$$

Varying (2) with respect to X^M and γ_{mn}, one obtains the equations of motion

$$\begin{aligned} &-g_{LK}\left[\partial_m\left(\sqrt{-\gamma}\gamma^{mn}\partial_n X^K\right) + \sqrt{-\gamma}\gamma^{mn}\Gamma^K_{MN}\partial_m X^M\partial_n X^N\right] \qquad (3)\\ &= \frac{1}{2}H_{LMN}\varepsilon^{mn}\partial_m X^M\partial_n X^N, \end{aligned}$$

and the constraints

$$\left(\gamma^{kl}\gamma^{mn} - 2\gamma^{km}\gamma^{ln}\right)\partial_m X^M\partial_n X^N g_{MN}(X) = 0, \qquad (4)$$

where

$$\Gamma^K_{MN} = g^{KL}\Gamma_{LMN} = \frac{1}{2}g^{KL}\left(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}\right)$$

is the symmetric connection compatible with the metric g_{MN} and

$$H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$$

is the B_{MN} field strength. The field $F_{\mu\nu}$ does not enter the equations of motion, because $dF \equiv 0$.

The field B'_{MN} explicitly appears in the expressions for the generalized momenta

$$P_M = -T\left(\sqrt{-\gamma}g_{MN}\gamma^{0n}\partial_n X^N - B'_{MN}\partial_1 X^N\right), \tag{5}$$

and in the boundary conditions

$$\left[\sqrt{-\gamma}g_{M\nu}\gamma^{1n}\partial_n X^\nu + B'_{M\nu}\partial_0 X^\nu\right]_{\sigma=0,\pi} = 0, \tag{6}$$
$$X^a(\tau,0) = X^a(\tau,\pi) = q^a, \qquad a = p+1, ..., D-1. \tag{7}$$

Here we have split the coordinates X^M into X^μ and X^a, and have denoted the location of the D-brane with q^a.

From now on, we will work in the gauge $\gamma^{mn} = constants$. We note that $\gamma^{mn} = \eta^{mn} = diag(-1,1)$ correspond to the commonly used *conformal gauge*.

2.1 Solutions of the Equations of Motion

First of all, we will try to find *background independent* solutions of the equations of motion of the type

$$X^M(\xi) = F^M(a_n\xi^n), \qquad a_n = constants.$$

It turns out that such solutions exist when $\gamma^{mn}a_m a_n = 0$. This leads to

$$X^M(\xi) = F^M_\pm(u_\pm), \qquad u_\pm = -\frac{1}{\gamma^{00}}\left(\gamma^{01} \pm \frac{1}{\sqrt{-\gamma}}\right)\xi^0 + \xi^1, \tag{8}$$

or

$$X^M(\xi) = F^M_\pm(v_\pm), \qquad v_\pm = \xi^0 + \frac{1}{\gamma^{11}}\left(-\gamma^{01} \pm \frac{1}{\sqrt{-\gamma}}\right)\xi^1, \tag{9}$$

where $F^M_\pm$ are arbitrary functions of their arguments. The main consequence of the obtained result is that for *arbitrary background fields* there exist only one solution, F^M_+ or F^M_-, but not both at the same time, in contrast with the flat space-time case. In other words, we have only chiral background independent solutions of the string equations of motion. On the other hand, it must be noted that these are not solutions of the constraints (4). Taking a linear combination of the two independent constraints, we can arrange one of them to be satisfied, but the other one will give restrictions on the metric. However, it can be shown that in the zero tension limit, the background independent solutions of the string equations of motion are also solutions of the corresponding constraints. Moreover, this result extends to arbitrary *tensionless* p-branes [9]. It

corresponds to the limit $(-\gamma)^{-1/2} \to 0$, taken in the expressions for $u_\pm$ and $v_\pm$. Let us also note that in conformal gauge, the obtained string solutions $F^M_\pm(u_\pm)$ and $F^M_\pm(v_\pm)$ reduce to the solutions $X^M_\pm(\sigma \pm \tau)$ and $X^M_\pm(\tau \pm \sigma)$ for left- or right-movers.

Our next step is to search for *non-chiral* solutions of the string equations of motion and constraints, i.e. solutions of the type

$$X^M(\xi^m) = F^M_+(w_+) + F^M_-(w_-), \tag{10}$$

where $w_\pm = u_\pm$ or $w_\pm = v_\pm$. Putting (10) in the equations of motion (3), we obtain the conditions for the existence of such solutions:

$$(2\Gamma_{L,MN} + H_{LMN}) \frac{dF^M_+}{dw_+}\frac{dF^N_-}{dw_-} = 0. \tag{11}$$

For simplicity, we will consider the case when g_{MN} and B_{MN} depend on only one coordinate, say r [1], and will give the results in conformal gauge.

In our first example, we fix all string coordinates X^M except X^0 and r (the remaining coordinates are denoted as X^α). Then the conditions (11) and constraints (4) reduce to the system of equations

$$\begin{aligned}
&\partial_r g_{00}[(\partial_0 X^0)^2 - (\partial_1 X^0)^2] - \partial_r g_{rr}[(\partial_0 r)^2 - (\partial_1 r)^2] = 0,\\
&\partial_r B_{0\alpha}(\partial_0 X^0 \partial_1 r - \partial_1 X^0 \partial_0 r) + \partial_r g_{0\alpha}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r)\\
&+\partial_r g_{r\alpha}[(\partial_0 r)^2 - (\partial_1 r)^2] = 0,\\
&\partial_r g_{00}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r) + \partial_r g_{0r}[(\partial_0 r)^2 - (\partial_1 r)^2] = 0,\\
&g_{00}[(\partial_0 X^0)^2 + (\partial_1 X^0)^2] + g_{rr}[(\partial_0 r)^2\\
&+(\partial_1 r)^2] + 2g_{0r}(\partial_0 X^0 \partial_0 r + \partial_1 X^0 \partial_1 r) = 0,\\
&g_{00}\partial_0 X^0 \partial_1 X^0 + g_{rr}\partial_0 r \partial_1 r + g_{0r}(\partial_0 X^0 \partial_1 r + \partial_1 X^0 \partial_0 r) = 0.
\end{aligned}$$

Among the nontrivial solutions of the above system, there exist the following non-chiral ones:

$$\begin{aligned}
&\frac{g_{rr}}{g_{00}} = \frac{\partial_r g_{rr}}{\partial_r g_{00}} = -\left(\frac{\partial_r g_{r\alpha}}{\partial_r B_{0\alpha}}\right)^2, \qquad g_{0r} = 0,\\
&\partial_r B_{0\alpha}\partial_0 X^0 = \partial_r g_{r\alpha}\partial_1 r, \qquad \partial_0 X^0 \partial_0 r = \partial_1 X^0 \partial_1 r;
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\frac{\partial_r g_{r\alpha}}{\partial_r g_{0\alpha}} = \frac{\partial_r g_{0r}}{\partial_r g_{00}}, \qquad \partial_r g_{rr} = \frac{(\partial_r g_{0r})^2}{\partial_r g_{00}}, \qquad g_{rr} = \left(2g_{0r} - g_{00}\frac{\partial_r g_{0r}}{\partial_r g_{00}}\right)\frac{\partial_r g_{0r}}{\partial_r g_{00}},\\
&\partial_r g_{00}\partial_0 X^0 = -\partial_r g_{0r}\partial_0 r, \qquad \partial_0 X^0 \partial_1 r = \partial_1 X^0 \partial_0 r;
\end{aligned} \tag{13}$$

[1] In the next subsection, the coordinate r will be associated with the radial coordinate in the directions transverse to the Dp-brane, on which the open string ends.

$$
\begin{aligned}
&\frac{\partial_r g_{00}}{g_{00}} = \frac{\partial_r g_{0r}}{g_{0r}} = \frac{\partial_r g_{rr}}{g_{rr}},\\
&-\left[\partial_r g_{0\alpha}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r) + \partial_r B_{0\alpha}(\partial_0 X^0 \partial_1 r - \partial_1 X^0 \partial_0 r)\right]/\partial_r g_{r\alpha}\\
&= -\partial_r g_{00}(\partial_0 X^0 \partial_0 r - \partial_1 X^0 \partial_1 r)/\partial_r g_{0r}\\
&= \partial_r g_{00}\left[(\partial_0 X^0)^2 - (\partial_1 X^0)^2\right]/\partial_r g_{rr}\\
&= (\partial_0 r)^2 - (\partial_1 r)^2.
\end{aligned}
\tag{14}
$$

In our second example, all string coordinates are kept fixed except X^0, X^1 and r. Only to have readable final expressions, we restrict the metric to be diagonal, and the NS-NS field to be constant. The solutions of the corresponding equations following from (11) and (4) are

$$
\begin{aligned}
&\partial_0 X^0 = +f\partial_1 r, \quad \partial_1 X^0 = +f\partial_0 r, \quad \partial_0 X^1 = +h\partial_1 r, \quad \partial_1 X^1 = +h\partial_0 r;\\
&\partial_0 X^0 = +f\partial_1 r, \quad \partial_1 X^0 = +f\partial_0 r, \quad \partial_0 X^1 = -h\partial_1 r, \quad \partial_1 X^1 = -h\partial_0 r;\\
&\partial_0 X^0 = -f\partial_1 r, \quad \partial_1 X^0 = -f\partial_0 r, \quad \partial_0 X^1 = +h\partial_1 r, \quad \partial_1 X^1 = +h\partial_0 r;\\
&\partial_0 X^0 = -f\partial_1 r, \quad \partial_1 X^0 = -f\partial_0 r, \quad \partial_0 X^1 = -h\partial_1 r, \quad \partial_1 X^1 = -h\partial_0 r,
\end{aligned}
$$

where

$$
\begin{aligned}
f(g,\partial g) &= \left(\frac{g_{rr}\partial_r g_{11} - g_{11}\partial_r g_{rr}}{g_{11}\partial_r g_{00} - g_{00}\partial_r g_{11}}\right)^{1/2},\\
h(g,\partial g) &= \left(\frac{g_{00}\partial_r g_{rr} - g_{rr}\partial_r g_{00}}{g_{11}\partial_r g_{00} - g_{00}\partial_r g_{11}}\right)^{1/2}.
\end{aligned}
$$

As a consequence, one receives from here that the following equalities are fulfilled

$$
(\partial_0 \pm \partial_1)X^0 = f(g,\partial g)(\partial_0 \pm \partial_1)r, \qquad (\partial_0 \pm \partial_1)X^1 = h(g,\partial g)(\partial_0 \pm \partial_1)r.
$$

Therefore, we have obtained solutions which allow for *all* string coordinates to be non-chiral.

2.2 Solving the Boundary Conditions

To be able to solve explicitly the boundary conditions, we assume that g_{MN} and B'_{MN} are constant at $\sigma = 0, \pi$. This is automatically achieved if g_{MN} and B_{MN} depend only on X^a, and the $U(1)$ field strength $F_{\mu\nu}$ is constant. These conditions are typically realized in string theory backgrounds, where g_{MN} and B_{MN} does not depend on the coordinates along the source brane, representing the exact solution of the *effective* string equations of motion, i.e. the equations of motion in the corresponding supergravity field theory [2, 3]. Moreover, B_{MN}

[2]Here the dilaton Φ does not appear explicitly, because we are working with the string frame metric, in which it is included.

[3]Thus, we implicitly imply that the Dp-brane, on which the open string ends, is parallel to the source (D)p'- brane, and $p' \geq p$.

often depends only on the radial coordinate r in the transverse to the source brane directions.

To find which non-chiral solutions of the probe string equations of motion and constraints give also a nontrivial solution of the boundary conditions (6) and (7), we first write down (10) in the form (compare with (9)):

$$X^M(\tau,\sigma) = X_+^M(\tau+\Sigma_+\sigma) + X_-^M(\tau+\Sigma_-\sigma), \qquad (15)$$
$$\Sigma_\pm \equiv \frac{1}{\gamma^{11}}\left(-\gamma^{01} \pm \frac{1}{\sqrt{-\gamma}}\right)$$

Using that the *background independent* solutions $X_\pm^M$ can be expanded as

$$X_\pm^M(\tau+\Sigma_\pm\sigma) = q_\pm^M + \alpha_{0\pm}^M(\tau+\Sigma_\pm\sigma) + i\sum_{k\neq 0}\frac{1}{k}\alpha_{k\pm}^M e^{-ik(\tau+\Sigma_\pm\sigma)},$$

we represent (15) in the form

$$X^M(\tau,\sigma) = q^M + a_0^M\left(\tau - \frac{\gamma^{01}}{\gamma^{11}}\sigma\right) + \frac{b_0^M\sigma}{\gamma^{11}\sqrt{-\gamma}} \qquad (16)$$
$$+\sum_{k\neq 0}\frac{e^{-ik\left(\tau-\frac{\gamma^{01}}{\gamma^{11}}\sigma\right)}}{k}\left[ia_k^M\cos\left(\frac{k\sigma}{\gamma^{11}\sqrt{-\gamma}}\right) + b_k^M\sin\left(\frac{k\sigma}{\gamma^{11}\sqrt{-\gamma}}\right)\right],$$

where

$$\alpha_{k\pm}^M = \frac{1}{2}\left(a_k^M \pm b_k^M\right).$$

Substituting (16) into the boundary conditions, we find the following solution of (6) and (7)

$$X^\mu(\tau,\sigma;\Sigma) = q^\mu + \left[\delta_\nu^\mu\left(\tau - \frac{\gamma^{01}}{\gamma^{11}}\sigma\right) - \left(g^{-1}B'\right)^\mu{}_\nu(q^a)\Sigma\sigma\right]a_0^\nu$$
$$+\sum_{k\neq 0}\frac{e^{-ik\left(\tau-\frac{\gamma^{01}}{\gamma^{11}}\sigma\right)}}{k}\left[i\delta_\nu^\mu\cos\left(k\Sigma\sigma\right) - \left(g^{-1}B'\right)^\mu{}_\nu(q^a)\sin\left(k\Sigma\sigma\right)\right]a_k^\nu, \qquad (17)$$
$$X^a(\tau,\sigma;\Sigma) = q^a + \sum_{k\neq 0}\frac{e^{-ik\left(\tau-\frac{\gamma^{01}}{\gamma^{11}}\sigma\right)}}{k}b_k^a\sin\left(k\Sigma\sigma\right),$$

where

$$\left(g^{-1}B'\right)^\mu{}_\nu = g^{\mu M}B'_{M\nu} = g^{\mu\rho}B'_{\rho\nu} + g^{\mu a}B'_{a\nu}, \qquad \Sigma = \frac{1}{\gamma^{11}\sqrt{-\gamma}} \in \mathbf{Z}_+.$$

Thus we have showed that there exist *exact* solutions of the equations of motion and constraints for the open string - Dp-brane system in *non-constant*

background fields, which are also *exact* solutions of the corresponding boundary conditions. For background metric and NS-NS two-form gauge field depending on the coordinates X^a, transverse to the Dp-brane, their explicit form is given by (17). This is achieved due to the existence of the exact string solutions $X^M_\pm(w_\pm)$, found in the previous subsection, which do not depend on the background fields.

Let us show on an example, that there exist solutions of the equation of motion (11) and of the constraints, which are also solutions of the boundary conditions of the type (17). To this end, consider the equations (12). They incorporate two types of conditions - on the background fields and on the string embedding coordinates. The solution for the background is

$$\begin{aligned}&ds^2 = g(r)\left[-a^2(dx^0)^2 + dr^2\right] + 2g_{0\alpha}(r)dx^0dx^\alpha\\ &+2g_{r\alpha}(r)drdx^\alpha + g_{\alpha\beta}(r)dx^\alpha dx^\beta,\\ &B_{0\alpha}(r) = b \pm a g_{r\alpha}(r),\end{aligned} \tag{18}$$

where a, b are arbitrary constants, and the metric coefficients are arbitrary functions of r. On (18), (12) reduces to

$$a\partial_0 X^0(\tau,\sigma) = \pm\partial_1 r(\tau,\sigma), \qquad a\partial_1 X^0(\tau,\sigma) = \pm\partial_0 r(\tau,\sigma). \tag{19}$$

Replacing (19) into (17), one obtains the following open string solution for the background fields (18)

$$\begin{aligned}X^0(\tau,\sigma) &= q^0 + i\sum_{k\neq 0}\frac{e^{-ik\tau}}{k}\cos(k\sigma)a^0_k,\\ X^1(\tau,\sigma) \equiv r(\tau,\sigma) &= q^r \pm a\sum_{k\neq 0}\frac{e^{-ik\tau}}{k}\sin(k\sigma)a^0_k.\end{aligned}$$

Finally, in order to establish the correspondence with the known solution of the equations of motion and of the boundary conditions in the case of constant background fields [10], we make the following restrictions:

1. $g_{MN} = g_{MN}(X^a), \quad B_{MN} = B_{MN}(X^a) \quad \to$
   $g_{MN} = constants, \quad B_{MN} = constants;$

2. $g_{MN}, \quad B_{MN}$ - arbitrary $\quad\to$
   $g_{MN} = \eta_{MN} = \mathrm{diag}(-,+,\ldots,+), \quad B_{\mu a} = 0, \quad B_{ab} = 0;$

3. worldsheet gauge:

$$\begin{aligned}\gamma^{mn} &= \text{arbitrary constants} \qquad \to\\ \gamma^{mn} = \eta^{mn} &= \mathrm{diag}(-,+) \qquad \Rightarrow \qquad \Sigma = 1.\end{aligned}$$

Under the above conditions, our solution (17) reduces to the one given in [10].

3 Proposal for a New Approach

It is clear that a crucial role in treating the open string - D-brane system in variable external fields is played by the conditions (11), which ensure the existence of nontrivial solutions of the type (10). Actually, (11) are the equations of motion for such type of string solutions. However, they do not contain second derivatives. That is why, we propose to consider them as *additional constraints* in the Hamiltonian description of the considered dynamical system. So, let us compute the resulting constraint algebra.

3.1 Poisson Brackets

Using the manifest expression (5) for the momenta, we obtain the following set of constraints ($\partial X \equiv \partial X/\partial\sigma$)

$$\begin{aligned} I_0 &\equiv g^{MN}P_MP_N - 2T\left(g^{-1}B'\right)^M{}_N P_M\partial X^N \\ &+T^2\left(g - B'g^{-1}B'\right)_{MN}\partial X^M\partial X^N, \end{aligned} \tag{20}$$

$$I_1 \equiv P_N\partial X^N - Tg_{MK}\left(g^{-1}B'\right)^K{}_N\partial X^M\partial X^N = P_N\partial X^N, \tag{21}$$

$$\begin{aligned} I_L &\equiv \Gamma_{L,MN}g^{MS}g^{NK}P_SP_K - T\left[2\Gamma_{L,MN}\left(g^{-1}B'\right)^N{}_K\right. \\ &\left.+ H_{LMK}\right]g^{MS}P_S\partial X^K \\ &+T^2\left\{\Gamma_{L,MN}\left[\left(g^{-1}B'\right)^M{}_S\left(g^{-1}B'\right)^N{}_K - \delta^M_S\delta^N_K\right]\right. \\ &\left.+ H_{LMS}\left(g^{-1}B'\right)^M{}_K\right\}\partial X^S\partial X^K. \end{aligned} \tag{22}$$

These constraints have one and the same structure. Namely, all of them are particular cases of the expression

$$\begin{aligned} I_J &\equiv K^{SK}_J(g,\partial g)P_SP_K + S^S_{JK}(g,\partial g,B',\partial B')P_S\partial X^K \\ &+R_{JSK}(g,\partial g,B',\partial B')\partial X^S\partial X^K, \end{aligned}$$

where $J = (n, L)$, and the coefficient functions K^{SK}_J, S^S_{JK} and R_{JSK} depend on X^N and do not depend on P_N. The computation of the Poisson brackets, assuming canonical ones for the coordinates and momenta, gives

$$\begin{aligned} \{I_{J_1}(\sigma_1), I_{J_2}(\sigma_2)\} &= \left[M^K_{(J_1}N_{J_2)K}(\sigma_1) + M^K_{(J_1}N_{J_2)K}(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2) \\ &+ C_{[J_1J_2]}\delta(\sigma_1-\sigma_2), \end{aligned} \tag{23}$$

where (J_1, J_2) and $[J_1, J_2]$ mean symmetrization and antisymmetrization in the indices J_1, J_2 respectively. Obviously, the algebra does not close on I_J. On the other hand, the right hand side is quadratic with respect to the newly appeared structures M^S_J and N_{JS}. They are given by

$$M^S_J = 2K^{SN}_J P_N + S^S_{JN}\partial X^N, \qquad N_{JS} = S^M_{JS}P_M + 2R_{JSM}\partial X^M,$$

and satisfy the following Poisson brackets among themselves

$$\left\{M_{J_1}^{S_1}(\sigma_1), M_{J_2}^{S_2}(\sigma_2)\right\} = \left[\left(K_{J_1}^{S_1N} S_{J_2N}^{S_2} + K_{J_2}^{S_2N} S_{J_1N}^{S_1}\right)(\sigma_1)\right.$$
$$\left.+\left(K_{J_1}^{S_1N} S_{J_2N}^{S_2} + K_{J_2}^{S_2N} S_{J_1N}^{S_1}\right)(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2) + C_{J_1J_2}^{S_1S_2}\delta(\sigma_1-\sigma_2),$$
$$\{N_{J_1S_1}(\sigma_1), N_{J_2S_2}(\sigma_2)\} = \left[\left(S_{J_1S_1}^{N} R_{J_2S_2N} + S_{J_2S_2}^{N} R_{J_1S_1N}\right)(\sigma_1)\right.$$
$$\left.+\left(S_{J_1S_1}^{N} R_{J_2S_2N} + S_{J_2S_2}^{N} R_{J_1S_1N}\right)(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2) + C_{J_1J_2S_1S_2}\delta(\sigma_1-\sigma_2),$$
$$\left\{M_{J_1}^{S_1}(\sigma_1), N_{J_2S_2}(\sigma_2)\right\} = \left[\left(2K_{J_1}^{S_1N} R_{J_2S_2N} + \frac{1}{2} S_{J_1N}^{S_1} S_{J_2S_2}^{N}\right)(\sigma_1)\right.$$
$$\left.+\left(2K_{J_1}^{S_1N} R_{J_2S_2N} + \frac{1}{2} S_{J_1N}^{S_1} S_{J_2S_2}^{N}\right)(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2) + C_{J_1J_2S_2}^{S_1}\delta(\sigma_1-\sigma_2).$$

M_J^S and N_{JS} act on P_M and ∂X^M as follows

$$\begin{aligned}
&\left\{M_J^S(\sigma_1), P_M(\sigma_2)\right\} = S_{JM}^S(\sigma_2)\partial\delta(\sigma_1-\sigma_2)\\
&+\left[2\partial_M K_J^{SN} P_N + \left(\partial_M S_{JN}^S - \partial_N S_{JM}^S\right)\partial X^N\right]\delta(\sigma_1-\sigma_2),\\
&\{N_{JS}(\sigma_1), P_M(\sigma_2)\} = 2R_{JSM}(\sigma_2)\partial\delta(\sigma_1-\sigma_2)\\
&+\left[\partial_M S_{JS}^N P_N + 2\left(\partial_M R_{JSN} - \partial_N R_{JSM}\right)\partial X^N\right]\delta(\sigma_1-\sigma_2),\\
&\left\{M_J^S(\sigma_1), \partial X^M(\sigma_2)\right\} = 2K_J^{SM}(\sigma_2)\partial\delta(\sigma_1-\sigma_2)\\
&-2\partial_N K_J^{SM}\partial X^N\delta(\sigma_1-\sigma_2),\\
&\left\{N_{JS}(\sigma_1), \partial X^M(\sigma_2)\right\} = S_{JS}^M(\sigma_2)\partial\delta(\sigma_1-\sigma_2) - \partial_N S_{JS}^M\partial X^N\delta(\sigma_1-\sigma_2).
\end{aligned}$$

Actually, I_J can be expressed in terms of M_J^S and N_{JS} as

$$I_J = \frac{1}{2}\left(M_J^K P_K + N_{JK}\partial X^K\right).$$

Let us now see how from the above *open* algebra the *closed* algebra of the constraints arises. For the gauge generators I_n, we have

$$\begin{aligned}
I_0: \quad & M_0^M = 2\left[g^{MN}P_N - T\left(g^{-1}B'\right)^M{}_N\partial X^N\right],\\
& N_{0M} = 2T\left[\left(B'g^{-1}\right)_M{}^N P_N + T\left(g - B'g^{-1}B'\right)_{MN}\partial X^N\right];\\
I_1: \quad & M_1^M = \partial X^M, \qquad N_{1M} = P_M.
\end{aligned}$$

Inserting these expressions in (23) one obtains

$$\begin{aligned}
\{I_0(\sigma_1), I_0(\sigma_2)\} &= (2T)^2\left[I_1(\sigma_1) + I_1(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2),\\
\{I_1(\sigma_1), I_1(\sigma_2)\} &= \left[I_1(\sigma_1) + I_1(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2),\\
\{I_0(\sigma_1), I_1(\sigma_2)\} &= \left[I_0(\sigma_1) + I_0(\sigma_2)\right]\partial\delta(\sigma_1-\sigma_2).
\end{aligned} \tag{24}$$

The equalities (24) reproduce the known result stating that the string constraint algebra in a gravitational and 2-form gauge field background coincides with the one in flat space-time [11].

The Poisson bracket between I_1 and I_L closes on I_L and is given by:

$$\{I_1(\sigma_1), I_L(\sigma_2)\} = [I_L(\sigma_1) + I_L(\sigma_2)]\, \partial\delta(\sigma_1 - \sigma_2).$$

The remaining brackets, $\{I_0(\sigma_1), I_L(\sigma_2)\}$ and $\{I_{L_1}(\sigma_1), I_{L_2}(\sigma_2)\}$, are of the general type (23).

3.2 Lagrangian Picture

In order to realize from the Lagrangian approach viewpoint what are the consequences of our idea to include another set of constraints in the string Hamiltonian for a *certain type of solutions*, now we are going to find the corresponding Lagrangian density. To be able to compare the results step by step with the usual case and clearly see the differences, we will first going through the procedure namely in this case.

We start with a Hamiltonian density, which is a linear combination of the constraints I_0 and I_1, given by (20) and (21) respectively

$$\mathcal{H} = \lambda^0 I_0 + \lambda^1 I_1.$$

Then we obtain the corresponding Hamiltonian equations of motion for the coordinates X^M

$$(\partial_0 - \lambda^1\partial_1)X^M = 2\lambda^0 g^{MN}\left(P_N - TB'_{NK}\partial_1 X^K\right).$$

From here, we express the momenta P_M as functions of $\partial_m X^N$

$$P_M = \frac{g_{MN}}{2\lambda^0}(\partial_0 - \lambda^1\partial_1)X^N + TB'_{MN}\partial_1 X^N. \tag{25}$$

By using (25), the constraints I_0 and I_1 can now be rewritten as

$$I_0 = \frac{1}{(2\lambda^0)^2} g_{MN}(\partial_0 - \lambda^1\partial_1)X^M(\partial_0 - \lambda^1\partial_1)X^N \tag{26}$$
$$+T^2 g_{MN}\partial_1 X^M\partial_1 X^N,$$
$$I_1 = \frac{g_{MN}}{2\lambda^0}(\partial_0 - \lambda^1\partial_1)X^M\partial_1 X^N. \tag{27}$$

The computation of the Lagrangian density gives

$$\begin{aligned} \mathcal{L} &= P_N\partial_0 X^N - \lambda^0 I_0 - \lambda^1 I_1 \qquad (28)\\ &= \frac{1}{4\lambda^0} g_{MN}(\partial_0 - \lambda^1\partial_1)X^M(\partial_0 - \lambda^1\partial_1)X^N \\ &\quad +TB'_{MN}\partial_0 X^M\partial_1 X^N - \lambda^0 T^2 g_{MN}\partial_1 X^M\partial_1 X^N. \end{aligned}$$

The equations of motion for λ^m, obtained from (28), give the constraints (26) and (27). The Euler-Lagrange equations for X^M, based on (28), are

$$\begin{aligned}
&g_{LN}(\partial_0-\lambda^1\partial_1)(\partial_0-\lambda^1\partial_1)X^N+\Gamma_{L,MN}(\partial_0-\lambda^1\partial_1)X^M(\partial_0-\lambda^1\partial_1)X^N\\
&-(2\lambda^0T)^2\left(g_{LN}\partial_1^2X^N+\Gamma_{L,MN}\partial_1X^M\partial_1X^N\right)\\
&-2\lambda^0TH_{LMN}\partial_0X^M\partial_1X^N=0.
\end{aligned}\tag{29}$$

Now we are interested in non-chiral solution of the above equations of motion and constraints of the type (10)

$$X^M(\xi^m)=X_+^M(v_+^\lambda)+X_-^M(v_-^\lambda),\tag{30}$$

where $X_\pm^M$ are the two background independent solutions of the equations (29), and $v_\pm^\lambda$ are the variables $v_\pm$ defined in (9), taken in λ-parametrization

$$v_\pm^\lambda=\tau+\frac{\sigma}{\lambda^1\pm2\lambda^0T}.$$

For this type of solutions, the equations of motion (29) acquire the form (11), with $w_\pm$ replaced with $v_\pm^\lambda$. With the help of the explicit expressions for the momenta (25), one can further transform these equations to obtain the expressions (22) for I_L, which we would like to include as additional constraints in the Hamiltonian, describing such type string solutions.

In order to realize this idea, we now begin with the Hamiltonian density

$$\mathcal{H}=\lambda^0I_0+\lambda^1I_1+\lambda^LI_L.$$

The corresponding Hamiltonian equations of motion for the coordinates X^M are

$$\begin{aligned}
(\partial_0-\lambda^1\partial_1)X^M&=2\lambda^0\left(\mathcal{M}^{-1}\right)^{MN}\left(P_N-TB'_{NK}\partial_1X^K\right)\\
&-T\lambda^Lg^{MN}H_{LNK}\partial_1X^K,
\end{aligned}$$

where we have introduced the notation

$$\left(\mathcal{M}^{-1}\right)^{MN}\equiv g^{MN}+\frac{\lambda^L}{\lambda^0}g^{MK}\Gamma_{L,KS}g^{SN}.\tag{31}$$

It follows from here that the momenta P_M, as functions of ∂_mX^N, are given by

$$P_M=\frac{\mathcal{M}_{MN}}{2\lambda^0}(\partial_0-\lambda^1\partial_1)X^N+T\mathcal{A}_{MN}\partial_1X^N,\tag{32}$$

where

$$\mathcal{A}_{MN}\equiv B'_{MN}+\frac{\lambda^L}{2\lambda^0}\left(\mathcal{M}g^{-1}\right)_M{}^KH_{LKN}.\tag{33}$$

Comparing (32) with the previous expressions for the momenta (25), we see that they both have the same form, but in (32) new, effective background fields

$\mathcal{M}_{MN}$ and $\mathcal{A}_{MN}$ have appeared. In view of (31) and (33), they have the form of a specific *deformation* of the initial background.

By using (32), the constraints I_0, I_1 and I_L can be rewritten as

$$I_0 = \frac{1}{(2\lambda^0)^2}\left(\mathcal{M}g^{-1}\mathcal{M}\right)_{MN}(\partial_0-\lambda^1\partial_1)X^M(\partial_0-\lambda^1\partial_1)X^N \tag{34}$$
$$+\frac{T}{\lambda^0}\left[\mathcal{M}g^{-1}(\mathcal{A}-B')\right]_{MN}\partial_0X^M\partial_1X^N$$
$$+T^2\left\{\left[g+(\mathcal{A}-B')\right]g^{-1}\left[g-(\mathcal{A}-B')\right]\right\}_{MN}\partial_1X^M\partial_1X^N,$$
$$I_1 = \frac{\mathcal{M}_{MN}}{2\lambda^0}(\partial_0-\lambda^1\partial_1)X^M\partial_1X^N, \tag{35}$$
$$I_L = \frac{1}{(2\lambda^0)^2}\left(\mathcal{M}g^{-1}\Gamma_Lg^{-1}\mathcal{M}\right)_{MN}(\partial_0-\lambda^1\partial_1)X^M(\partial_0-\lambda^1\partial_1)X^N \tag{36}$$
$$+\frac{T}{\lambda^0}\left\{\mathcal{M}g^{-1}\left[\Gamma_Lg^{-1}(\mathcal{A}-B')-\frac{1}{2}H_L\right]\right\}_{MN}\partial_0X^M\partial_1X^N$$
$$-T^2\left\{(\mathcal{A}-B')g^{-1}\left[\Gamma_Lg^{-1}(\mathcal{A}-B')-H_L\right]+\Gamma_L\right\}_{MN}\partial_1X^M\partial_1X^N,$$

where

$$(\Gamma_L)_{MN}\equiv\Gamma_{L,MN}\quad\text{and}\quad(H_L)_{MN}\equiv H_{LMN}.$$

The computation of the corresponding Lagrangian density gives

$$\begin{aligned}\mathcal{L}^{new} &= P_N\partial_0X^N-\lambda^0I_0-\lambda^1I_1-\lambda^LI_L \\ &= \frac{1}{4\lambda^0}\mathcal{M}_{MN}(\partial_0-\lambda^1\partial_1)X^M(\partial_0-\lambda^1\partial_1)X^N \\ &\quad +T\mathcal{A}_{MN}\partial_0X^M\partial_1X^N-\lambda^0T^2\Delta_{MN}\partial_1X^M\partial_1X^N,\end{aligned} \tag{37}$$

where

$$\Delta_{MN}\equiv\left\{2g-\left[g+(\mathcal{A}-B')\right]\mathcal{M}^{-1}\left[g-(\mathcal{A}-B')\right]\right\}_{MN}.$$

Comparing (37) with (28), we see that both $\mathcal{L}$ and $\mathcal{L}^{new}$ are of the same form, but in (37) we have new, effective antisymmetric background field $\mathcal{A}_{MN}$, and *two* new, effective background "metrics" - $\mathcal{M}_{MN}$ and Δ_{MN}.

The equations of motion for the Lagrange multipliers λ^m, λ^L, obtained from (37), give the constraints (34), (35) and (36). The Euler-Lagrange equations for X^M, following from (37), are

$$\begin{aligned}&\mathcal{M}_{LN}(\partial_0-\lambda^1\partial_1)(\partial_0-\lambda^1\partial_1)X^N \\ &+\Gamma^{\mathcal{M}}_{L,MN}(\partial_0-\lambda^1\partial_1)X^M(\partial_0-\lambda^1\partial_1)X^N \\ &-(2\lambda^0T)^2\left(\Delta_{LN}\partial_1^2X^N+\Gamma^{\Delta}_{L,MN}\partial_1X^M\partial_1X^N\right) \\ &-2\lambda^0TH^{\mathcal{A}}_{LMN}\partial_0X^M\partial_1X^N=0,\end{aligned} \tag{38}$$

where the notations used are given by the equalities

$$\Gamma^{\mathcal{M}}_{L,MN}\equiv\frac{1}{2}\left(\partial_M\mathcal{M}_{NL}+\partial_N\mathcal{M}_{ML}-\partial_L\mathcal{M}_{MN}\right),$$

$$\Gamma^{\Delta}_{L,MN} \equiv \frac{1}{2}\left(\partial_M \Delta_{NL} + \partial_N \Delta_{ML} - \partial_L \Delta_{MN}\right),$$
$$H^{\mathcal{A}}_{LMN} \equiv \partial_L \mathcal{A}_{MN} + \partial_M \mathcal{A}_{NL} + \partial_N \mathcal{A}_{LM}.$$

As far as we are interested in non-chiral solutions of the above equations of motion (see (30)), we insert (30) in (38) and obtain

$$\begin{aligned}
&(\mathcal{M}_{LN} - \Delta_{LN})\left[\left(\lambda^1 - 2\lambda^0 T\right)^2 \frac{d^2 X^N_+}{d(v^\lambda_+)^2} + \left(\lambda^1 + 2\lambda^0 T\right)^2 \frac{d^2 X^N_-}{d(v^\lambda_-)^2}\right] \\
&+ \left(\Gamma^{\mathcal{M}}_{L,MN} - \Gamma^{\Delta}_{L,MN}\right)\left[\left(\lambda^1 - 2\lambda^0 T\right)^2 \frac{dX^M_+}{dv^\lambda_+}\frac{dX^N_+}{dv^\lambda_+}\right. \\
&\left. + \left(\lambda^1 + 2\lambda^0 T\right)^2 \frac{dX^M_-}{dv^\lambda_-}\frac{dX^N_-}{dv^\lambda_-}\right] \\
&-2\left(\lambda^1 - 2\lambda^0 T\right)\left(\lambda^1 + 2\lambda^0 T\right)\left(\Gamma^{\mathcal{M}}_{L,MN} + \Gamma^{\Delta}_{L,MN} + H^{\mathcal{A}}_{LMN}\right)\frac{dX^M_+}{dv^\lambda_+}\frac{dX^N_-}{dv^\lambda_-} = 0.
\end{aligned} \tag{39}$$

Comparing (11) with (39), we see that the latter acquires the form of the former only when the two "metrics", $\mathcal{M}_{MN}$ and Δ_{MN}, coincide.

Since part of the Hamiltonian constraints are actually our original Euler-Lagrange equations for X^M on the specified class of solutions, we must give new interpretation of the equations (39), different from viewing them as equations of motion. An important observation in this direction is that the equations (39) contain second derivatives of the original background fields. Therefore, they can be connected to the *second variation* of the (original) action (2) and consequently, to the *sufficient conditions* for existence of a "local" minimum of the action on the extremal surfaces. This problem deserves separate consideration and will be studied elsewhere.

4 Concluding Remarks

In this paper we considered the case of an open string ending on D-brane in variable external fields $g_{MN}(x)$ and $B_{MN}(x)$, in the framework of the sigma-model approach. In Sec.2 we formulated the problem and obtained exact solutions of the string equations of motion, constraints and boundary conditions of a particular type. In the constant background fields limit, our solutions reduce to a *generalization* of the result already known [10].

In Sec.3, we investigate some of the consequences of the idea that the conditions for existence of non-chiral solutions of the string equations of motion in variable gravitational and NS-NS fields, can be reinterpreted as additional constraints. In particular, we compute the corresponding Poisson bracket algebra, which in the most general case does not close on the initial constraints, but is quadratic with respect to *two* newly appeared structures. The classical Virasoro algebra does not changes and appears here as a subalgebra. In the last part of Sec.3, we analyze the changes due to the inclusion of the new constraints

from the Lagrangian approach point of view. It turns out that this results in a specific *deformation* of the initial background.

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